The Psychopathology of Factor Indeterminancy

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To cite this article: Peter H. Schonemann (1996) The Psychopathology of Factor Indeterminancy, Multivariate Behavioral Research, 31:4, 571-577, DOI: 10.1207/s15327906mbr3104_10

To link to this article: http://dx.doi.org/10.1207/s15327906mbr3104_10

Published online: 10 Jun 2010.

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The Psychopathology of Factor Indeterminacy

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At any given moment there is an orthodoxy, a body of ideas, which it is assumed that all right-thinking people will accept without question ... Anyone who challenges the prevailing orthodoxy finds himself silenced with surprising effectiveness. (George Orwell, as cited in Thorpe, 1978, p. 6).

Factor Indeterminacy

The issue of factor indeterminacy has intermittently been a topic for debate ever after E.B. Wilson first uncovered it 70 years ago. Conscious of the limitations of his audience, Wilson (1928/1981) spelled it out in very simple terms: “Try a case. Let the marks of 6 students on 3 tests be ...” (p. 377). He then proceeded to show, with explicit reference to 18 numbers, that the factors of Spearman’s (1904) factor model, including \( g \), are not uniquely determined by the equations of his Two Factor Theory. The basic reason is simply that Spearman’s model postulates more factors than tests. After a decade of lively discussion, the problem was quietly shelved. So far as I know, neither Thurstone nor Harman ever addressed it explicitly during the decades of “exploratory factor analysis”. Half a century later, Guttman tried his hand at explaining a simple problem simply:

It may help to make the point clear by considering the following “true score” problem. Suppose it is known that for a certain variable \( T \), individual \( i \) has one and only one value \( T_i \), and this value satisfies

\[
T_i^2 - 100T_i + 99 = 0
\]  

What is individual \( i \)'s score on \( T \)?

The answer, of course, is that \( T_i \) is either 99 or 1; the statement of the problem leaves the actual value indeterminate in the sense that too many solutions exist to condition (1)” (Guttman, personal communication to Green, cited in Steiger and Schönenmann, 1978, p. 137).

The author is on leave from Purdue University. Support of the Foundation for the Advancement of Outstanding Scholarship is gratefully acknowledged.
On first blush it may seem that the basic point at issue is simple enough, to be understood by sophomores. Maraun (1996), however, wrestles with the much more difficult question why most psychometricians, and some statisticians of world reknown — for example Mallows and Tukey (1982, p. 149) who speak of a “presumed indeterminacy of factor scores” — have insisted on acting as if it were beyond their grasp.

Maraun’s (1996) central thesis is that this was because they confused an elementary mathematical problem with the extra-mathematical “metaphor” that had inspired it (viz. Spearman’s, 1904, attempt to define “intelligence” as a common factor, \( g \)). This confounding of form with substance obscured the elementary nature of the issue.

Positions and Changing Positions

In his first draft, Maraun (1996) partitioned the set of scholars who have demonstrated some interest in the indeterminacy into two subsets: (a) those adopting an Alternative Solution Position (ASP, e.g., Wilson, Camp, Guttman, Steiger, Wang, & myself), and (b) those adopting a Posterior Moment Position (PMP, e.g., McDonald, Bartholomew, & Holland). Responding to reviewer’s criticisms, he added a third, Infinite Behavior Domain Position which he credits to Williams (1978), and also to McDonald, who has repeatedly revised his position over the years.

In the end, Maraun (1996) comes down on the side of the Alternative Solution Position. This strikes me as sensible.

The Posterior Moment Position

I have never been able to muster much enthusiasm for the Posterior Moment Position, though for different reasons than Maraun (1996). Just as Wilson (1928/1981) and Guttman (personal communication to Green, cited in Steiger and Schönenmann, 1978, p. 137) had demonstrated with their simple numerical illustrations, factor indeterminacy is an algebraic problem which has little to do with Borel sets or the difficulties some psychologists may have with the notion of random variables. Wilson, who first noticed the indeterminacy after psychologists and statisticians alike had overlooked it for 25 years, said that much from the very beginning: “...it would seem quite superfluous to introduce this higher mathematics, involving a probability theory which probably does not apply anyhow, to make determinate (if it does) that which without it seems indeterminate” (Wilson, 1928/1981, p. 377).

The indeterminacy has little to do with random variables because it arises in all inner product spaces (vector spaces equipped with a scalar...
product so that length and angle can be measured). Scalar random variables satisfy the formal vector space requirements, because, as real-valued functions, they are evidently closed under linear combinations over the reals. But so are subsets of many other functions, for example, polynomials over the reals, that have nothing to do with random variables. If we are given two polynomials, \( y_1, y_2 \), as functions of three unknown polynomials, \( x, z_1, z_2 \), and are told that the latter relate to the former by the equations of the factor model, then we will not be able to uniquely solve for \( x, z_1, z_2 \) either because the system of equations is indeterminate (not: "presumed indeterminate").

The same is true for score \( N \)-tuples which, as elements of \( \mathbb{R}^N \), also form a vector space. All relevant derivations can be carried out equivalently "in the sample" (i.e., on score matrices) or "in the population" (on vectors of random variables) with minor changes in notation. In the population, the only relevant role the distributions play is to provide the inner product.

Far from resolving the indeterminacy, contemplation of conditional distributions actually renders it more apparent: in factor analysis, we start out with \( p \) tests, and wind up with \( p + m \) factors, if the model fits. Suppose it does fit and we compute the partial covariance matrix \( \text{Var}(x|y) \) (i.e., the \( v - c \) matrix of the conditional distribution of the \( m \) common factors \( x \), given the tests \( y \)). The factor model implies that this matrix will be non-singular. Think about it: after removing all the information we started with, we are left with \( m \) additional linearly independent variables. Where did they come from?

These, and many other oddities, are consequences of the indeterminacy that have nothing to do with the fact that the vectors happen to be random variables. For example, in (Schönemann & Steiger, 1978) we showed that "... the factor model implies the existence of criteria that, although perfectly correlated with the observed scores \( Y \) (in a multiple regression sense), remain completely unpredictable from the common factor scores ... [and it] also implies the existence of criteria that, although entirely uncorrelated with the observed scores \( Y \), are positively correlated with suitably defined common factor scores \( X' \)" (p. 289. See Steiger, 1979, for more on external validity). On invoking a different scalar product (by changing "perfectly uncorrelated" to "perfectly orthogonal" etc.), the same startling results apply to vector spaces of real polynomials, for example.

**Are Factors Random Variables?**

Strictly speaking, the factors of the factor model cannot be random variables anyway. Random variables are usually defined as maps of
probability spaces into the (sometimes: product) space of the reals, and maps
as many-one relations. Now, while the relation from the sample space to the
test space is many-one, and thus a map, the relation from the test space to the
factor space is, in view of the indeterminacy, many-many and thus not a
map. Hence, the composite relation from the sample space to the factor
space is not a map either, and factors cannot be random variables by the
conventional definitions. This is just another way of saying that the
homomorphism from the vectorspace of the factors to that of the tests is not
invertible. However, compared with other, more urgent problems of the
factor model, this minor technicality hardly seems worth worrying about.

The Infinite Behavior Domain Position

I am somewhat more favorably inclined than Maraun (1996) towards the
"Infinite Behavior Domain Position", albeit only in theory.

It is true that most authors, myself included, have discussed the
indeterminacy in terms of a finite number of tests. In this case one is stuck
with an irreparable indeterminacy which, as we showed empirically for the
first time in (Schönemann & Wang, 1972), is much more severe and
widespread in practice than its long neglect may have suggested. It gets
worse as one extracts or postulates more factors as does, for example, the
"LISREL model". The indeterminacy afflicts modern Item Response
Theory no less than classical True Score Theory (which is simply a special
case of Spearman's model). As Wechsler (1939), with his own inimitable
logic, had observed long ago: "As to the existence of "g" as a comon factor,
there seems to be no possibility of doubt. Psychometrics without it, loses its
basic prop." (p. 8).

Thus, although simple enough as a mathematical problem, the
indeterminacy is fraught with implications for psychology generally. It also
beclouds the efforts of experts on "factor score estimation". Their first line
of defense was to dismiss it as "trivial" (cf. Guttman, 1975/1994, p. 368).
After this expedient had proven untenable, Williams (1978, p. 305) entered
the fray, announcing that "... no adequate model has ever been set out before,
only finite-dimensional factor analysis equations have been studied".
Actually, Guttman had dwelled at length on the infinite variable case
following Piaggio's (1933) treatment of the single common factor case.

What is more, a cogent case can be made that Spearman (1904) himself
— though unaware of the indeterminacy until 1928 — must have had the
infinite domain metaphor in mind when he proposed his Two Factor Theory
in 1904: He wanted to define General Intelligence, g, as the sole common
factor explaining all cognitive performance measures. His theory predicted
that, no matter which battery was analyzed, it should always satisfy the tetrad difference condition (and thus imply exactly one common factor, namely g). That he failed to formalize this more global model explicitly is less important than the fact that his metaphor clearly implied that the addition of new variables should preserve rank 1. If this had been borne out empirically, then the indeterminacy would indeed have vanished in the limit, although, as Mulaik and McDonald (1978) have pointed out, the limit would not have been unique but would have depended on which variables were added. This is a minor complication — though well worth noting — compared with a much more serious problem:

The Important Difference Between Facts And Fictions

Williams' (1978) model can be viewed as a belated formalization of Spearman's (1904) metaphor encompassing the infinite domain. The problem with this elegant "elimination of problems of factor score indeterminacy" (p. 293) is that the necessary condition — that the rank stay constant as more variables are added — is never met by any data: as one adds more tests one invariably winds up with more factors (usually by a factor of 1/3). This was well-known in the 30s and 40s and became a cornerstone of Thurstone's (1947) Multiple Factor Analysis dogma. His disciples (e.g., McNemar, 1951) chided Spearman for his naivite in assuming just one General Intelligence factor.

Williams' (1978) explicit formalization of the Infinite Domain thesis — that there must exist in the real world a unique factor — nicely illustrates what is wrong with it: The same method could be used to eliminate the need for umbrellas by constructing a model based on the assumption it will never rain again.

Spearman's Hypothesis

A minor blemish of Maraun's (1996) article are repeated references to "Spearman's hypothesis" with the intended meaning of "Two Factor Theory". Recently this term has acquired a new meaning (cf. Jensen, 1985, Guttman, 1992; Schonemann, 1992).

Nevertheless, there are some interesting parallels. This latest paradigm shift disinters Spearman's (1904) g and ties it to Black/White differences, thus serving as a timely reminder that problems surrounding factor theory did and do have far-reaching social consequences. Maraun (1996) rightly criticizes the widespread malpractice of confusing regression weights with factor variables. In the past, such practices could have been dismissed as
inconsequential. In the context of Spearman’s Hypothesis in Jensen’s (1985) sense, they assume more ominous significance because they are employed in the service of a factorial theory of intelligence with a distinctly racist edge.

Jensen (1985) and his followers, while intoning Spearman’s g, skirt all conceptual complexities of the factor model by simply extracting first principal components, which they call “g”. This sleight of hand eliminates all falsifiability problems because there always must be a first principal component. As evidence that first principal components from different studies are the same g, they point out that the regression weights are all positive. This advanced case of exploratory data analysis culminates in Spearman’s Hypothesis in the modern sense, the claim that Blacks are genetically inferior to Whites because the mean differences between both groups correlate with the magnitude of the loadings on “g” (actually: on the first principal component).

Judging from the history of factor indeterminacy, and the traditional reluctance of statisticians to touch socially relevant statistical problems — except to obfuscate them further — it may take yet another 100 years before this latest perversion of Spearman’s factor model is sorted out.

Conclusions

I welcome Maraun’s (1996) incisive analysis of the various “positions” on factor indeterminacy and agree with most of his conclusions.

References


